# ON $d$-DIMENSIONAL $d$-SEMIMETRICS AND SIMPLEX-TYPE INEQUALITIES FOR HIGH-DIMENSIONAL SINE FUNCTIONS

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Abstract. We show that high-dimensional analogues of the sine function (more precisely, the $d$-dimensional polar sine and the $d$-th root of the $d$-dimensional hypersine) satisfy a simplex-type inequality in a real pre-Hilbert space $H$. Adopting the language of Deza and Rosenberg, we say that these $d$-dimensional sine functions are $d$-semimetrics. We also establish geometric identities for both the $d$-dimensional polar sine and the $d$-dimensional hypersine. We then show that when $d = 1$ the underlying functional equation of the corresponding identity characterizes a generalized sine function. Finally, we show that the $d$-dimensional polar sine satisfies a relaxed simplex inequality of two controlling terms "with high probability".

## 1. Introduction

We establish some fundamental properties of high-dimensional sine functions [9]. In particular, we show that they satisfy a simplex-type inequality, and thus according to the terminology of Deza and Rosenberg [7] they are $d$-semimetrics. We also demonstrate a related concentration inequality. These properties are useful to some modern investigations in harmonic analysis [14, 15] and applied mathematics [5, 6].

High-dimensional sine functions have been known for more than a century. Euler [10] formulated the two-dimensional polar sine (for tetrahedra) and D'Ovidio [8] generalized it to higher dimensions. Joachimsthal [11] suggested the two-dimensional hypersine (for tetrahedra) and Bartoš [4] extended it to simplices of any dimension. Various authors have explored their properties and applied them to a variety of problems (see e.g., [9], [22], [13], [23], [12] and references in there). For our purposes, we have slightly modified the existing definitions, in particular we allow negative values of these functions when the dimension of the ambient space is $d + 1$.

The two high-dimensional sine functions that we define here, $\mathrm{p}_d\mathrm{sin}$ and $\mathrm{g}_d\mathrm{sin}$, are exemplified in Figure 1 and are described as follows. For $v_1, \ldots, v_{d+1}, w \in H$ we take the parallelotope through the points $v_1, \ldots, v_{d+1}, w$. The function $|\,\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})|$ is obtained by dividing the $(d + 1)$-volume of that parallelotope by the $d + 1$ edge lengths at the vertex $w$. Similarly, we define $|\,\mathrm{g}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})|$ to be the $(d + 1)$-volume of the same parallelotope scaled by the $d$-th roots of the $d$-volumes of its faces through the vertex $w$ (there are $d + 1$ of these). That is, $|\,\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})|$ and $|\,\mathrm{g}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})|$ are the polar sine [9] and the $d$-th root of the hypersine [20] of the simplex with vertices $\{w, v_1, \ldots, v_{d+1}\}$ with respect to the vertex $w$. If $\dim(H) = d + 1$, then we define $\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})$ and $\mathrm{g}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})$ by replacing the volume of the parallelotope by the corresponding determinant (precise definitions appear in Subsection 2.4). We often assume that $w = 0$ since the more general case can be obtained by a simple shift (as expressed later in equations (7) and (8)).

We note that when $d = 1$: $|\,\mathrm{p}_1\mathrm{sin}_0(v_1, v_2)| = |g_1 \sin_0(v_1, v_2)| = |\sin(v_1, v_2)|$, where $|\sin(v_1, v_2)|$ denotes the absolute value of the sine of the angle between $v_1$ and $v_2$. Furthermore, regardless of the dimension of $H$, the following triangle inequality holds

$$|\sin(v_1, v_2)| \leq |\sin(v_1, u)| + |\sin(u, v_2)|, \text{ for all } v_1, v_2 \in H \text{ and } u \in H \setminus \{0\}. \tag{1}$$

The first part of this paper establishes high-dimensional analogues of equation (1) for the functions $|\,\mathrm{p}_d\mathrm{sin}\,|$ and $|\,\mathrm{g}_d\mathrm{sin}\,|$ for $d > 1$.

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One motivation for our research is the interest in high-dimensional versions of metrics and $d$-way kernel methods in machine learning [3, 21]. Deza and Rosenberg [7] have defined the notion of a $d$-semimetric (or $n$-semimetric according to their notation). If $d \in \mathbb{N}$ and $E$ is a given set, then the pair $(E, f)$ is a $d$-semimetric if $f : E^{d+1} \mapsto [0, \infty)$ is symmetric (invariant to permutations) and satisfies the following simplex-type inequality:

$$f(x_1, \ldots, x_{d+1}) \leq \sum_{i=1}^{d+1} f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{d+1}) \quad \text{for all} \quad x_1, \ldots, x_{d+1}, u \in E. \tag{2}$$

We also refer to $f$ itself as a $d$-semimetric with respect to $E$ or just a $d$-semimetric when the set E is clear.

The examples of $d$-semimetrics proposed by Deza and Rosenberg [7] do not represent $d$-dimensional geometric properties. They typically form $d$-semimetrics by averaging non-negative functions that quantify lower order geometric properties of a $d$-simplex (see [7, Fact 2]). For example, in order to form a 2-semimetric on $H$ they average the pairwise distances between three points to get the scaled perimeter of the corresponding triangle, which is a one-dimensional quantity.

We provide here the following $d$-dimensional examples of $d$-semimetrics.

**Theorem 1.1.** *If $H$ is a real pre-Hilbert space, $d \in \mathbb{N}$, and $\dim(H) \geq d + 1$, then the functions $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ are $d$-semimetrics with respect to the set $H \setminus \{0\}$.*

The above examples of $d$-semimetrics are $d$-dimensional in the following sense: $|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})|$ and $|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})|$ are zero if and only if the vectors $v_1, \ldots, v_{d+1}$ are linearly dependent, and they are one (and maximal) if and only if the vectors $v_1, \ldots, v_{d+1}$ are mutually orthogonal.

Another motivation for our research is our interest in high-dimensional generalizations of the Menger curvature [16, 18]. In a subsequent work [14, 15] we define a $d$-dimensional Menger-type curvature for $d > 1$ via the polar sine, $|\,\mathrm{p}_d\mathrm{sin}\,|$, and use it to characterize the smoothness of $d$-dimensional Ahlfors regular measures (see Definition 5.1). Our proof utilizes the fact that the polar sine satisfies "a relaxed simplex inequality of two controlling terms with high Ahlfors probability". We quantify this notion in a somewhat general setting as follows.

For a symmetric function $f$ on $H^{d+1}$, an integer $p$, $1 \leq p \leq d$, and a positive constant $C$, we say that $f$ satisfies a *relaxed simplex inequality* of $p$ terms and constant $C$ if

$$f(v_1, \ldots, v_{d+1}) \leq C \cdot \sum_{i=1}^{p} f(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}), \text{ for all } v_1, \ldots, v_{d+1} \in H \text{ and } u \in H \setminus \{0\}. \tag{3}$$

By the symmetry of $f$, any $p$ terms in the above sum will suffice (e.g., replacing $\sum_{i=1}^{p}$ by $\sum_{i=d+2-p}^{d+1}$ in equation (3)).

For any $S = \{v_1, \ldots, v_{d+1}\} \subseteq H$, $w \in H$, and $C > 0$, we let $U_C(S, w)$ be the set of vectors $u$ giving rise to relaxed simplex inequalities of two terms and constant $C$ for $|\,\mathrm{p}_d\mathrm{sin}_w\,|$, that is,

$$\begin{aligned} U_C(S, w) = \Big\{ u \in H : \ & |\,\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})| \leq C \cdot \big( |\,\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})| \, + \\ & |\,\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_{d+1})| \big), \text{ for all } 1 \leq i < j \leq d+1 \Big\}. \end{aligned} \tag{4}$$

Using this notation, we claim that for any $d$-dimensional Ahlfors regular measure $\mu$ on $H$, any sufficiently large constant $C$, any set of vectors $S$ as above and any $w \in \mathrm{supp}(\mu)$, the event $U_C(S, w)$ has high probability at any relevant ball in $H$, where a probability at a ball is obtained by scaling the measure $\mu$ by the measure of the ball. We formulate this property more precisely and even more generally as follows:

**Theorem 1.2.** *If $H$ is a pre-Hilbert space, $2 \leq d \in \mathbb{N}$, $0 < \epsilon < 1$, $\gamma \in \mathbb{R}$ is such that $d - 1 < \gamma \leq d$, $w \in \mathrm{supp}(\mu)$, $S = \{v_1, \ldots, v_{d+1}\} \subseteq H$, and $\mu$ is a $\gamma$-dimensional Ahlfors regular measure on $H$ with Ahlfors regularity constant $C_\mu$, then there exists a constant $C_0 \geq 1$ depending only on $C_\mu$, $\epsilon$, $\gamma$, and $d$, such that for all $C \geq C_0$:*

$$\frac{\mu\left(U_C(S, w) \cap B(w, r)\right)}{\mu\left(B(w, r)\right)} \geq 1 - \epsilon, \quad \text{for all } 0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu)). \tag{5}$$

The paper is organized as follows. In Section 2 we present the main notation and definitions as well as a few elementary properties of the $d$-dimensional sine functions. In Section 3 we develop geometric identities

for $\mathrm{p}_d\sin_0$ and $\mathrm{g}_d\sin_0$ as well as characterize the solutions of the corresponding functional equations when $d = 1$. In Section 4 we prove Theorem 1.1, and in Section 5 we prove Theorem 1.2. Finally, we conclude our research in Section 6 and discuss future directions and open problems.

## 2. Notation, Definitions, and Elementary Propositions

Our analysis takes place on a real pre-Hilbert space $H$, with an inner product denoted by $\langle\cdot,\cdot\rangle$. We denote by $\dim(H)$ the dimension of $H$, possibly infinite, and we often denote subspaces of $H$ by $V$ or $W$. The orthogonal complement of $V$ is denoted by $V^{\perp}$. If $V$ is a complete subspace of $H$ (in particular finite dimensional), then we denote the orthogonal projection of $H$ onto $V$ by $P_V$. We denote the norm induced by the inner product on $H$ by $\|\cdot\|$, and the distance between $x, y \in H$ by $\mathrm{dist}(x, y)$ or equivalently $\|x - y\|$. Similarly, $\mathrm{dist}(x, V) = \|P_V(x) - x\|$ is the induced distance between $x \in H$ and a complete subspace $V \subseteq H$.

We have chosen to work in the general setting of a pre-Hilbert space in order to emphasize the independence of our current and subsequent results [14, 15] from the dimension of the ambient space.

By $d$ we denote an intrinsic dimension of interest to us, where $d \in \mathbb{N}$ and $d + 1 \leq \dim(H)$. We also use the integer $k \geq 1$ according to our purposes. Whenever we use $d$ or $k$ and do not specify their range, one can always assume that they are positive integers and $k, d + 1 \leq \dim(H)$.

If $f$ is defined on $H^k$, then we denote the evaluation of $f$ on the ordered set of vectors $v_1, \ldots, v_{k+1} \in H$ with $v_j$ removed by $f(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k+1})$. We remark that we maintain this notation for all $1 \leq j \leq k$, in particular, $j = 1$ and $j = k + 1$. Similarly, for $1 \leq j \leq k$, then $f(v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_k)$ is $f$ evaluated on the ordered set of $k$ vectors $v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_k \in H$, where $v_j$ is replaced by $u$. We may remove two vectors, $v_i$ and $v_j$, from the ordered set $\{v_1, \ldots, v_{k+2}\}$ and denote the function $f$ evaluated on the resulting set by $f(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k+2})$, regardless of the order of $i$ and $j$ and whether or not either is 1 or $k + 2$. In this case the convention is always that $i \neq j$.

For an arbitrary subset $K$ in $H$, we denote its diameter by $\mathrm{diam}(K)$. If $\mu$ is a measure on $H$, we denote the support of $\mu$ by $\mathrm{supp}(\mu)$.

We follow with specific definitions and corresponding propositions according to topics.

### 2.1. Special Subsets of H.

For an affine subspace $L \subseteq H$, a point $x \in L$, and an angle $\theta$ such that $0 \leq \theta \leq \pi/2$, we define the *cone*, $\mathrm{C_{one}}(\theta, L, x)$, centered at $x$ on $L$ in the following way

$$\mathrm{C_{one}}(\theta, L, x) := \{u \in H : \mathrm{dist}(u, L) \leq \|u - x\| \cdot \sin(\theta)\}.$$

For an affine subspace, $L \subseteq H$ and $h > 0$, we define the *tube* of height $h$ on $L$, $\mathrm{T_{ube}}(L, h)$, as follows.

$$\mathrm{T_{ube}}(L, h) := \{u \in H : \mathrm{dist}(u, L) \leq h\}.$$

For $r > 0$ and $x \in H$, we define the *ball* of radius $r$ on $x$ to be

$$B(x, r) := \{u \in H : \|u - x\| \leq r\}.$$

### 2.2. Sets Generated by Vectors.

If $v_1 \ldots, v_k \in H$, then the *parallelotope* spanned by these vectors is the set

$$\mathrm{P_{rll}}(v_1, \ldots, v_k) := \left\{\sum_{i=1}^{k} t_i v_i \ : 0 \leq t_i \leq 1,\ i = 1, \ldots, k\right\}.$$

Similarly, the *polyhedral cone* spanned by $v_1, \ldots, v_k$ has the form

$$\mathrm{C_{poly}}(v_1, \ldots, v_k) := \left\{\sum_{i=1}^{k} t_i v_i \ : t_i \geq 0,\ i = 1, \ldots, k\right\}.$$

The *affine plane* through the vectors $v_1, \ldots, v_k$ is defined by

$$\mathrm{A_{ffn}}(v_1, \ldots, v_k) := \left\{\sum_{i=1}^{k} t_i v_i \ :\ \sum_{i=1}^{k} t_i = 1,\ \ t_i \in \mathbb{R},\ i = 1, \ldots, k\right\}.$$

The *convex hull* of $v_1, \ldots, v_k$ is the set

$$\mathrm{C_{hull}}(v_1, \ldots, v_k) := \mathrm{A_{ffn}}(v_1 \ldots, v_k) \cap \mathrm{C_{poly}}(v_1, \ldots, v_k).$$

If $S$ is a finite subset of $H$, we denote the span of $S$ by $L_S$, and sometimes also by $\mathrm{Sp}(S)$.

2.3. **Determinants and Contents.** If $H$ is finite-dimensional, $\dim(H) = k$, and $\Phi = \{\phi_1, \ldots, \phi_k\}$ is an arbitrary orthonormal basis for $H$, then we denote by $\det_\Phi$ the determinant function with respect to $\Phi$, that is, the unique alternating multilinear function such that $\det_\Phi(\phi_1, \ldots, \phi_k) = 1$. The following elementary property of the determinant will be fundamental in part of our analysis and hence we distinguish it.

**Proposition 2.1.** *If $\dim(H) = k$, $v_1, \ldots, v_k \in H$ and $u \in \mathrm{A_{ffn}}(v_1, \ldots, v_k)$, then for any orthonormal basis $\Phi$*

$$\det{}_\Phi(v_1, \ldots, v_k) = \sum_{i=1}^{k} \det{}_\Phi(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_k).$$

The arbitrary choice of $\Phi$ will not matter to us and thus will not be specified. Indeed, our major statements will involve only $|\det_\Phi|$, or will be related to Proposition 2.1, both of which are invariant under any choice of orthonormal basis $\Phi$. For this reason we will usually refer to "the determinant" and dispense with the subscript $\Phi$, i.e., $\det \equiv \det_\Phi$.

If $v_1, \ldots, v_k \in H$, we define the $k$-content of the parallelotope $\mathrm{P_{rll}}(v_1, \ldots, v_k)$, denoted by $M_k(v_1, \ldots, v_k)$, as follows:

$$M_k(v_1, \ldots, v_k) := \begin{cases} \det_\Phi(v_1, \ldots, v_k), & \text{if } k = \dim(H) \text{ for fixed } \Phi, \\ \left[\det\left(\{\langle v_i, v_j\rangle\}_{i,j=1}^{k}\right)\right]^{\frac{1}{2}}, & \text{if } k < \dim(H). \end{cases} \tag{6}$$

We note that if $k = \dim(H)$, then the $k$-content may obtain negative values, and that the absolute value of the $k$-content can be expressed by the same formula for all $k \leq \dim(H)$, i.e.,

$$|M_k(v_1, \ldots, v_k)| = \left[\det\left(\{\langle v_i, v_j\rangle\}_{i,j=1}^{k}\right)\right]^{\frac{1}{2}}.$$

2.4. **High-Dimensional Sine Functions.** Using the definition of $M_k$ in equation (6) and the Euclidean norm on $H$, we define the functions $\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})$ and $\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})$ respectively as

$$\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) := \frac{M_{d+1}(v_1, \ldots v_{d+1})}{\left(\prod_{j=1}^{d+1} M_d(v_1, \ldots v_{j-1}, v_{j+1} \ldots v_{d+1})\right)^{1/d}}$$

and

$$\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) := \frac{M_{d+1}(v_1, \ldots, v_{d+1})}{\prod_{j=1}^{d+1} \|v_j\|},$$

where if either of the denominators above is zero (and thus the numerator as well), then the corresponding function also obtains the value zero. We note that in the case of $d = 1$, both functions are essentially the ordinary sine functions. We exemplify this definition in Figure 1.

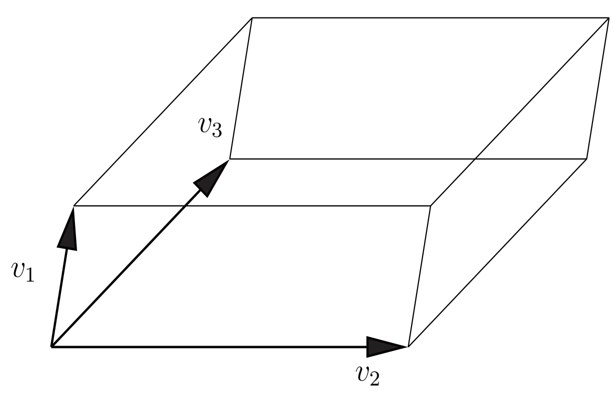


FIGURE 1. Exemplifying the computation of $\mathrm{p}_d\mathrm{sin}_0(v_1, v_2, v_3)$ and $\mathrm{g}_d\mathrm{sin}_0(v_1, v_2, v_3)$, when $d = 2$ and $H = \mathbb{R}^3$: The figure shows the parallelepiped spanned by $v_1$, $v_2$ and $v_3$. In this case, $\mathrm{p}_d\mathrm{sin}_0(v_1, v_2, v_3) = \dfrac{v_1 \bullet (v_2 \times v_3)}{\|v_1\| \cdot \|v_2\| \cdot \|v_3\|}$ and $\mathrm{g}_d\mathrm{sin}_0(v_1, v_2, v_3) = \dfrac{v_1 \bullet (v_2 \times v_3)}{\sqrt{\|v_1 \times v_2\| \cdot \|v_2 \times v_3\| \cdot \|v_1 \times v_3\|}}$.

For these functions and their vector arguments $v_1, \ldots, v_{d+1}$, we treat the point 0 as a distinguished vertex of the $(d+1)$-simplex through the vertices $\{0, v_1, \ldots, v_{d+1}\}$. More generally, we may add a vertex $w \in H$ other than 0, and we define the functions $\mathrm{g}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})$ and $\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})$ for vectors $v_1, \ldots, v_{d+1}$, $w \in H$ as follows:

$$\mathrm{g}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1}) = \mathrm{g}_d\mathrm{sin}_0(v_1 - w, \ldots, v_{d+1} - w), \tag{7}$$

and

$$\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1}) = \mathrm{p}_d\mathrm{sin}_0(v_1 - w, \ldots, v_{d+1} - w). \tag{8}$$

Whenever possible we refer to the functions $|\,\mathrm{p}_d\mathrm{sin}_w\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_w\,|$ so that we do not need to distinguish between the cases $\dim(H) = d+1$ and $\dim(H) > d+1$. We mainly use the notation $\mathrm{p}_d\mathrm{sin}_w$ or $\mathrm{g}_d\mathrm{sin}_w$ when $\dim(H) = d+1$. In particular, we may use the absolute values even if it is clear that $\dim(H) > d+1$ and thus the two sine functions are nonnegative.

We frequently use the following elementary property of $\mathrm{p}_d\mathrm{sin}_w$ and $\mathrm{g}_d\mathrm{sin}_w$, whose proof is included in Appendix A.1.

**Proposition 2.2.** *The functions* $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ *and* $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ *defined on* $H^{d+1}$ *are invariant under orthogonal transformations of* $H$ *and non-zero dilations of their arguments. Moreover, if* $\dim(H) = d+1$*, then* $\mathrm{p}_d\mathrm{sin}_0$ *and* $\mathrm{g}_d\mathrm{sin}_0$ *are invariant under dilations by positive coefficients.*

Finally, we describe a generalized law of sines for $\mathrm{g}_d\mathrm{sin}$ following Eriksson [9] (see also Bartoš [4]):

**Proposition 2.3.** *If* $\{0, v_1, \ldots, v_{d+1}\} \subseteq H$ *are vertices of a non-degenerate* $(d+1)$*-simplex, then for all* $1 \leq i \neq j \leq d+1$*:*

$$\frac{|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})|^d}{M_d(v_1 - v_{d+1}, \ldots, v_d - v_{d+1})} =$$
$$\frac{|\,\mathrm{g}_d\mathrm{sin}_{v_i}(v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_{d+1})|^d}{M_d(v_1 - v_j, \ldots, v_{j-1} - v_j, v_{j+1} - v_j, \ldots, v_{i-1} - v_j, -v_j, v_{i+1} - v_j, \ldots, v_{d+1} - v_j)}.$$

The proof follows from the definition of $|\,\mathrm{g}_d\mathrm{sin}_0\,|$. A reformulation of this law is the invariance of the function $|\,\mathrm{g}_d\mathrm{sin}_u(v_1, \ldots, v_{d+1})|/M_d(v_1 - u, \ldots, v_{d+1} - u)^{1/d}$ with respect to permuting its arguments, $u$ included.

2.5. **Elevation, Maximal Elevation, and Dihedral Angles.** For a complete and non-trivial subspace $W \subseteq H$ and $u \in H \setminus \{0\}$, we define the *elevation angle* of $u$ with respect to $W$ to be the smallest angle that $u$ makes with any element $w \in W \setminus \{0\}$, and we denote this angle by $\theta(u, W)$. More formally, in this case

$$\theta(u, W) = \min_{w \in W \setminus \{0\}} \left\{ \arccos\left( \left\langle \frac{u}{\|u\|}, \frac{w}{\|w\|} \right\rangle \right) \right\}.$$

If $u = 0$, then we take $\theta(0, W) = 0$. We call the sines of these angles *elevation sines* and note the following formula for computing them:

$$\sin(\theta(u, W)) = \frac{\mathrm{dist}(u, W)}{\|u\|}. \tag{9}$$

If $V$ is a complete subspace of $H$ and $v_1, v_2 \in H$, we define the *maximal elevation angle* of $v_1$ and $v_2$ with respect to $V$, denoted by $\Theta(v_1, v_2, V)$, as follows:

$$\Theta(v_1, v_2, V) = \max\{\theta(v_1, V), \theta(v_2, V)\}. \tag{10}$$

Given finite dimensional subspaces $W$ and $V$ of $H$ such that $\dim(W) = \dim(V)$ and $\dim(W \cap V) = \dim(W) - 1$, we define the *dihedral angle* between $W$ and $V$ along $W \cap V$ to be the acute angle between the normals of $W \cap V$ in $W$ and $V$. We denote this angle by $\alpha(W, V)$. We call the sines of such angles *dihedral sines* and note the following formula for computing them:

$$\sin(\alpha(W, V)) = \frac{\mathrm{dist}(w, V)}{\mathrm{dist}(w, W \cap V)} = \frac{\mathrm{dist}(v, W)}{\mathrm{dist}(v, W \cap V)}, \quad \text{for all } w \in W \setminus V \text{ and } v \in V \setminus W. \tag{11}$$

2.6. **Product Formulas for the High-Dimensional Sine Functions.** Two of the most useful properties of the high-dimensional sine functions are their decompositions as products of lower-dimensional sines. For $v_1, \ldots, v_{d+1} \in H$ and $S = \{v_1, \ldots, v_{d+1}\}$, we formulate those decompositions as follows.

**Proposition 2.4.** $|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})|^d = \left(\prod_{i=1}^d \sin\left(\alpha\left(L_{S\setminus\{v_{d+1}\}}, L_{S\setminus\{v_i\}}\right)\right)\right) \cdot |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_d)|^{d-1}$.

**Proposition 2.5.** $|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| = \sin\left(\theta\left(v_{d+1}, L_{S\setminus\{v_{d+1}\}}\right)\right) \cdot |\,\mathrm{p}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_d)|$.

Proposition 2.4 was established in [9, equation 7], and Proposition 2.5 can be established given the fact that

$$
\begin{aligned}
|M_{d+1}(v_1, \ldots, v_{d+1})| &= \mathrm{dist}(v_{d+1}, L_{S\setminus\{v_{d+1}\}}) \cdot M_d(v_1, \ldots, v_d) \\
&= \|v_{d+1}\| \cdot \sin\left(\theta\left(v_{d+1}, L_{S\setminus\{v_{d+1}\}}\right)\right) \cdot M_d(v_1, \ldots, v_d).
\end{aligned} \tag{12}
$$

## 3. Functional Identities for High-Dimensional Sine Functions

Throughout this section we assume that $\dim(H) = d + 1$ and formulate identities for $\mathrm{p}_d\mathrm{sin}$ and $\mathrm{g}_d\mathrm{sin}$. We denote the vectors used for the arguments of the latter functions by $u$, $v_1, \ldots, v_{d+1} \in H$, and assume the following: $\{v_1, \ldots, v_{d+1}\}$ is a basis for $H$, $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$, and $u$ is not a scalar multiple of any of the individual basis vectors $v_1, \ldots, v_{d+1}$, in particular, $u \neq 0$.

The main elements of our identities are exemplified in Figure 2 and described as follows. We introduce positive free parameters $\{\beta_i\}_{i=1}^{d+1}$, and we note that $\mathrm{C}_{\mathrm{poly}}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$. We express the vector $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$ as a linear combination of $\{\beta_i v_i\}_{i=1}^{d+1}$ with coefficients $\{\lambda_i\}_{i=1}^{d+1}$, that is,

$$u = \sum_{i=1}^{d+1} \lambda_i \cdot \beta_i v_i\,. \tag{13}$$

We note that since $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$ and $u \neq 0$, we have that $\sum_{i=1}^{d+1} \lambda_i > 0$. We then define

$$\tilde{u} := \left(\sum_{i=1}^{d+1} \lambda_i\right)^{-1} u, \tag{14}$$

and observe that

$$\tilde{u} \in \mathrm{A}_{\mathrm{ffn}}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}). \tag{15}$$

Finally, Proposition 2.1 gives the fundamental identity used to establish all of the following identities:

$$\det(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \sum_{i=1}^{d+1} \det(\beta_1 v_1, \ldots, \beta_{i-1} v_{i-1}, \tilde{u}, \beta_{i+1} v_{i+1}, \ldots, \beta_{d+1} v_{d+1}). \tag{16}$$

In Subsection 3.1 we develop identities for $\mathrm{p}_d\mathrm{sin}_0$ by direct application of the above equations. Similarly, in Subsection 3.2 we develop identities for $\mathrm{g}_d\mathrm{sin}_0$ following the same equations. If $d = 1$, both identities for $\mathrm{p}_d\mathrm{sin}_0$ and $\mathrm{g}_d\mathrm{sin}_0$ reduce to a functional equation satisfied by the sine function. We characterize the general Lebesgue measurable solutions of the corresponding equation in Subsection 3.3.

3.1. **Identities for $\mathbf{p_d sin_0}$.** Dividing both sides of equation (16) by $\prod_{i=1}^{d+1} \|\beta_i v_i\|$, we obtain

$$\mathrm{p}_d\mathrm{sin}_0(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \sum_{i=1}^{d+1} P_i \cdot \mathrm{p}_d\mathrm{sin}_0(\beta_1 v_1, \ldots, \beta_{i-1} v_{i-1}, \tilde{u}, \beta_{i+1} v_{i+1}, \ldots, \beta_{d+1} v_{d+1}),$$

where

$$P_i \equiv P_i\left(\{\beta_i\}_{i=1}^{d+1},\, \{v_i\}_{i=1}^{d+1},\, u\right) = \frac{\|\tilde{u}\|}{\|\beta_i v_i\|}. \tag{17}$$

Applying either the law of sines or the formal definition of $\mathrm{p}_1\mathrm{sin}$, we express the coefficients $P_i$ as follows:

$$P_i = \frac{\mathrm{p}_1\mathrm{sin}_0(-\beta_i v_i, \tilde{u} - \beta_i v_i)}{\mathrm{p}_1\mathrm{sin}_0(\tilde{u}, -\tilde{u} + \beta_i v_i)}\,. \tag{18}$$

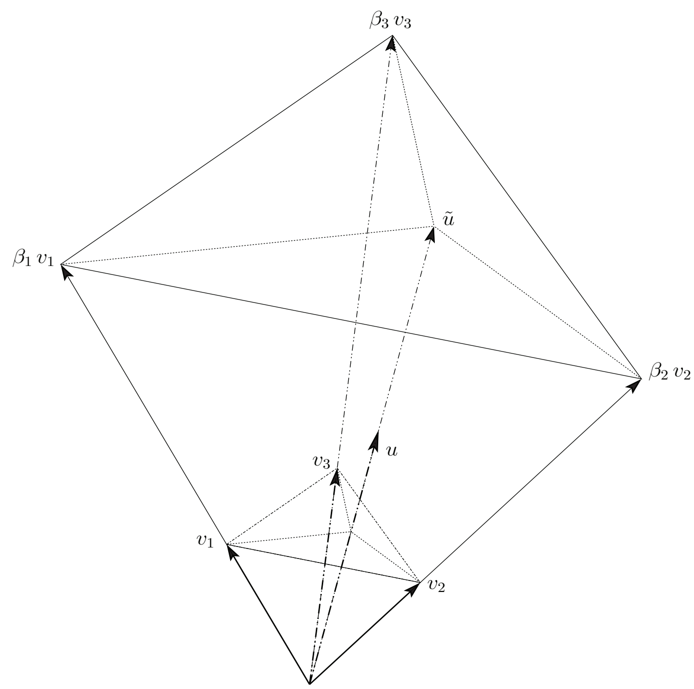


FIGURE 2. Exemplifying the basic construction of this section, when $d = 2$ and $H = \mathbb{R}^3$: We plot four particular vectors $v_1$, $v_2$, $v_3$ and $u$ and note that in this special case $u$ is not contained in the affine plane spanned by $v_1$, $v_2$ and $v_3$. We scale the latter three vectors arbitrarily by the positive parameters $\beta_1$, $\beta_2$ and $\beta_3$ and plot the resulting vectors. We form $\tilde{u}$ by scaling $u$ so that it is in the affine plane spanned by $\beta_1 v_1$, $\beta_2 v_2$ and $\beta_3 v_3$.

By the positive scale-invariance of $\mathrm{p}_d\mathrm{sin}_0$ we obtain that

$$\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) = \sum_{i=1}^{d+1} P_i \cdot \mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}). \tag{19}$$

By choosing different coefficients $\{\beta_i\}_{i=1}^{d+1}$ we can obtain different identities for $\mathrm{p}_d\mathrm{sin}_0$. There are only $d$ degrees of freedom in forming such identities due to the restriction of equation (16). In Subsection 4.2 we will use the following choice of $\{\beta_i\}_{i=1}^{d+1}$:

$$\beta_i = \frac{1}{\|v_i\|}, \; i = 1, \ldots, d+1. \tag{20}$$

The coefficients $\{P_i\}_{i=1}^{d+1}$, as described in equation (17), thus obtain the form,

$$P_1 = \ldots = P_{d+1} = \|\tilde{u}\| \tag{21}$$

and consequently equation (19) becomes

$$\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) = \|\tilde{u}\| \cdot \sum_{i=1}^{d+1} \mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}). \tag{22}$$

At last we exemplify the above identities when $d = 1$. We denote the angle between $v_1$ and $u$ by $\alpha > 0$, and the angle between $u$ and $v_2$ by $\beta > 0$, so that $\alpha + \beta$ is the angle between $v_1$ and $v_2$. We note that by the two assumptions of linear independence and $u \in \mathrm{C}_{\mathrm{poly}}(v_1, v_2)$ we have that $\alpha + \beta < \pi$. We denote the angle between $-u$ and $v_1 - u$ by $\delta$, where $\beta < \delta < \pi - \alpha$. The parameter $\delta$ represents the unique degree of freedom.

In this case, equations (18) and (19) reduce to the following trigonometric identity:

$$\sin(\alpha + \beta) = \frac{\sin(\alpha + \delta)}{\sin(\delta)} \cdot \sin(\beta) + \frac{\sin(\delta - \beta)}{\sin(\delta)} \cdot \sin(\alpha). \tag{23}$$

This identity generalizes to all $\alpha$, $\beta \in \mathbb{R}$ and $\delta \in \mathbb{R} \setminus \pi\mathbb{Z}$. It was used in [17] and is also very natural when establishing Ptolemy's theorem by trigonometry.

Furthermore, equation (22) reduces to the trigonometric identity

$$\sin(\alpha+\beta) = \frac{\sin(\frac{\alpha+\beta}{2})}{\sin(\frac{\alpha-\beta}{2})} \cdot \big(\sin(\alpha) - \sin(\beta)\big),$$

which can also be derived from equation (23) by setting $\delta = (\beta - \alpha)/2$.

3.2. **Identities for $\mathbf{g_d sin_0}$.** We now establish similar identities for $\mathrm{g}_d\mathrm{sin}_0$. Dividing both sides of equation (16) by $\prod_{j=1}^{d+1} \left(M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})\right)^{1/d}$ we obtain that

$$\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) = \sum_{i=1}^{d+1} Q_i \cdot \mathrm{g}_d\mathrm{sin}_0(\beta_1 v_1, \ldots, \beta_{i-1}v_{i-1}, \tilde{u}, \beta_{i+1}v_{i+1}, \ldots, \beta_{d+1}v_{d+1}), \tag{24}$$

where

$$Q_i = \left( \prod_{\substack{j=1\\ j\neq i}}^{d+1} \frac{M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{i-1}v_{i-1}, \tilde{u}, \beta_{i+1}v_{i+1}, \ldots, \beta_{d+1}v_{d+1})}{M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})} \right)^{1/d}. \tag{25}$$

By the positive scale-invariance of $\mathrm{g}_d\mathrm{sin}_0$, we rewrite equation (24) as

$$\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) = \sum_{i=1}^{d+1} Q_i \cdot \mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}). \tag{26}$$

We can express the coefficients $Q_i$ in different ways. First, we note that

$$Q_i = \prod_{\substack{j=1\\ j\neq i}}^{d+1} \frac{\mathrm{g}_d\mathrm{sin}_{\beta_i v_i}(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, 0, \beta_{j+1}v_{j+1}, \ldots, \beta_{i-1}v_{i-1}, \tilde{u}, \beta_{i+1}v_{i+1}, \ldots, \beta_{d+1}v_{d+1})}{\mathrm{g}_d\mathrm{sin}_{\tilde{u}}(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, 0, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})}. \tag{27}$$

The fact that the absolute values of both equations (25) and (27) are the same follows from the generalized law of sines (see Proposition 2.3). Moreover, the terms $\{Q_i\}_{i=1}^{d+1}$ in equation (27) are positive (see Appendix A.2), as are the corresponding terms of equation (25).

A different expression for $\{Q_i\}_{i=1}^{d+1}$ can be obtained as follows. We set $S = \{v_1, \ldots, v_{d+1}\}$ and notice that equation (12) implies that for all $1 \leq i < j \leq d+1$ :

$$\frac{M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{i-1}v_{i-1}, \tilde{u}, \beta_{i+1}v_{i+1}, \ldots, \beta_{d+1}v_{d+1})}{M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})} = \frac{\mathrm{dist}\left(\tilde{u}, L_{S\setminus\{v_i,v_j\}}\right)}{\mathrm{dist}\left(\beta_i v_i, L_{S\setminus\{v_i,v_j\}}\right)}.$$

Therefore, the coefficients $Q_i$, $i = 1, \ldots, d+1$, have the form

$$Q_i = \prod_{\substack{j=1\\ j\neq i}}^{d+1} \left( \frac{\mathrm{dist}\left(\tilde{u}, L_{S\setminus\{v_i,v_j\}}\right)}{\mathrm{dist}\left(\beta_i v_i, L_{S\setminus\{v_i,v_j\}}\right)} \right)^{1/d}. \tag{28}$$

By further application of equation (9), we obtain that

$$Q_i = \frac{\|\tilde{u}\|}{\|\beta_i v_i\|} \cdot \prod_{\substack{j=1\\ j\neq i}}^{d+1} \left( \frac{\sin\left(\theta\left(\tilde{u}, L_{S\setminus\{v_i,v_j\}}\right)\right)}{\sin\left(\theta\left(\beta_i v_i, L_{S\setminus\{v_i,v_j\}}\right)\right)} \right)^{1/d}. \tag{29}$$

It thus follows from equations (17) and (29) that

$$Q_i = P_i \cdot \prod_{\substack{j=1\\ j\neq i}}^{d+1} \left( \frac{\sin\left(\theta\left(\tilde{u}, L_{S\setminus\{v_i,v_j\}}\right)\right)}{\sin\left(\theta\left(\beta_i v_i, L_{S\setminus\{v_i,v_j\}}\right)\right)} \right)^{1/d}. \tag{30}$$

There are different possible choices for the parameters $\{\beta_i\}_{i=1}^{d+1}$, and we present a specific choice and its consequence in Subsection 4.1.1.

3.3. **Characterization of the Solutions of the One-Dimensional Identity.** When $d = 1$, the identities of Subsections 3.1 and 3.2 can be reduced to equation (23). That is, $f(x) = \sin(x)$ satisfies the functional equation

$$f(\alpha+\beta) = \frac{f(\alpha+\delta)}{f(\delta)} \cdot f(\beta) + \frac{f(\delta-\beta)}{f(\delta)} \cdot f(\alpha) \quad \text{for all } \alpha, \beta \in \mathbb{R},\ \delta \in \mathbb{R} \setminus f^{-1}(0)\,. \tag{31}$$

We show here that the most general Lebesgue measurable solutions of equation (31) are multiples of the generalized sine functions on spaces of constant curvature [12], i.e., functions of the form $c \cdot s_k(x)$, where

$$s_k(x) = \begin{cases} \frac{\sin(\sqrt{k}x)}{\sqrt{k}}, & \text{if } k > 0, \\ x, & \text{if } k = 0, \\ \frac{\sinh(\sqrt{-k}x)}{\sqrt{-k}}, & \text{otherwise.} \end{cases}$$

We remark that equation (31) is almost identical to an equation suggested by Mohlenkamp and Monzón [17, equation (5)], but has a different set of solutions. It is also closely related to Carmichael's equation [1, Section 2.5.2, equation (1)] as the proof of the following proposition shows.

We denote the set of multiples of generalized sine functions on spaces of constant curvature by $\mathcal{S}$, that is,

$$\mathcal{S} = \left\{ c \cdot s_k(x) : \ c, k \in \mathbb{R} \right\}.$$

Using this notation, we formulate the main result of this subsection:

**Theorem 3.1.** *The set of all Lebesgue measurable functions satisfying equation (31) coincides with $\mathcal{S}$.*

*Proof.* Clearly the elements of $\mathcal{S}$ satisfy equation (31). We thus assume that $f$ is a Lebesgue measurable function satisfying equation (31) and show that $f \in \mathcal{S}$. We denote the set of zeros of $f$ by $f^{-1}(0)$. Since $f = 0$ is an element of $\mathcal{S}$ (obtained by setting $c = 0$ in equation (32)), we also assume that $f \neq 0$ and in particular $\mathbb{R} \setminus f^{-1}(0)$ is not empty.

We first observe that $f(0) = 0$. Indeed, by arbitrarily fixing $\delta \in \mathbb{R} \setminus f^{-1}(0)$ and setting $\alpha = -\beta$ in equation (31) we obtain that

$$f(0) = \frac{f(\delta-\beta)}{f(\delta)} \left( f(\beta) + f(-\beta) \right). \tag{32}$$

Setting also $\beta = 0$, we get that $f(0) = 0$.

Next, we show that the set $f^{-1}(0)$ has measure zero. We first note that it is closed under addition. Indeed, if $\alpha,\ \beta \in f^{-1}(0)$, then equation (31) implies that $f(\alpha + \beta) = 0$. Now, assuming that $f^{-1}(0)$ has positive measure and applying a classical result of Steinhaus [2, Theorem 6], we obtain that $f^{-1}(0)$, equivalently $f^{-1}(0) + f^{-1}(0)$, contains an open interval. Then, if 0 is an accumulation point of $f^{-1}(0)$, by the additivity of zeros such an open interval extends to $\mathbb{R}$, that is, $f^{-1}(0) = \mathbb{R}$, which is the case we excluded ($f = 0$). Consequently, either $f^{-1}(0)$ has measure zero, or we must have that 0 is not an accumulation point of $f^{-1}(0)$ and $f^{-1}(0)$ contains an open interval. However, this latter case results in a contradiction as we show next.

Setting $\beta = \gamma \in \mathbb{R} \setminus f^{-1}(0)$, $\delta \in \mathbb{R} \setminus f^{-1}(0)$, and $\alpha = \lambda \in f^{-1}(0)$ in equation (31), we get the formal relation:

$$\frac{f(\gamma+\lambda)}{f(\gamma)} = \frac{f(\delta+\lambda)}{f(\delta)}\,.$$

We thus conclude that for any $\lambda \in f^{-1}(0)$ there exists a constant $C(\lambda) \in \mathbb{R} \setminus \{0\}$ such that

$$f(\gamma+\lambda) = C(\lambda) \cdot f(\gamma) \quad \text{for all } \gamma \in \mathbb{R}\,. \tag{33}$$

This equation implies that the case where 0 is not an accumulation point of $f^{-1}(0)$ and $f^{-1}(0)$ contains an open interval cannot exist. We therefore conclude that $f^{-1}(0)$ has measure zero.

Using the fact that $f^{-1}(0)$ is a null set and combining it with equation (32), we can show that $f$ is an odd function. Indeed, if $f$ is not odd, then there exists a $\beta \in \mathbb{R}$ such that

$$f(\beta) + f(-\beta) \neq 0\,,$$

and by equation (32) we have that $f(\delta - \beta) = 0$ for all $\delta \in \mathbb{R} \setminus f^{-1}(0)$. Hence,

$$\mathbb{R} \setminus f^{-1}(0) \,-\, \beta \subseteq f^{-1}(0)\,,$$

however this set inequality contradicts the fact that $f^{-1}(0)$ is null. Therefore, $f$ is odd.

We next observe that

$$\text{if } \lambda \in f^{-1}(0)\,, \text{ then } |f(\gamma+\lambda)| = |f(\gamma)| \text{ for all } \gamma \in \mathbb{R}\,. \tag{34}$$

Indeed, fixing $\lambda \in f^{-1}(0)$ and replacing $\gamma$ with $\gamma - \lambda$ in equation (33), we have that

$$f(\gamma) = C(\lambda) \cdot f(\gamma - \lambda) \text{ for all } \gamma \in \mathbb{R}\,.$$

Also, replacing $\gamma$ with $-\gamma$ in equation (33) and using the fact that $f$ is odd, we obtain that

$$f(\gamma - \lambda) = C(\lambda) \cdot f(\gamma) \text{ for all } \gamma \in \mathbb{R}\,.$$

The above two equations imply that $|C(\lambda)| = 1$ for all $\lambda \in f^{-1}(0)$ (the case $C(\lambda) = 0$ is excluded by the assumption that $f \neq 0$). Equation (34) then follows from equation (33).

At last, setting $\delta = \beta - \alpha$ in equation (31) and using the fact that $f$ is odd, we get that

$$f(\alpha+\beta) = \frac{f(\beta)}{f(\beta-\alpha)} \cdot f(\beta) - \frac{f(\alpha)}{f(\beta-\alpha)} \cdot f(\alpha) \text{ for all } \alpha,\, \beta \in \mathbb{R} \text{ such that } \beta - \alpha \in \mathbb{R} \setminus f^{-1}(0)\,. \tag{35}$$

Moreover, setting $\lambda = \alpha - \beta$ and $\gamma = \beta$ in equation (34), we obtain that

$$|f(\alpha)| = |f(\beta)| \text{ for all } \alpha,\, \beta \in \mathbb{R} \text{ such that } \alpha - \beta \in f^{-1}(0)\,. \tag{36}$$

Equations (35) and (36) imply that $f$ satisfies Carmichael's equation, i.e.,

$$f(\alpha+\beta) \cdot f(\beta-\alpha) = f(\beta)^2 - f(\alpha)^2.$$

Since the Lebesgue measurable solutions of this equation are the elements of $\mathcal{S}$ (see e.g., [2, Corollary 15]), we conclude that $f \in \mathcal{S}$. $\square$

## 4. Simplex Inequalities for High-Dimensional Sine Functions

In this section we prove Theorem 1.1, that is, we show that the functions $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ are $d$-semimetrics. We establish it separately for each of the functions in the following theorems.

**Theorem 4.1.** *If $v_1, \ldots, v_{d+1} \in H$ and $u \in H \setminus \{0\}$, then*

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|.$$

**Theorem 4.2.** *If $v_1, \ldots, v_{d+1} \in H$ and $u \in H \setminus \{0\}$, then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|.$$

The proofs of both theorems are parallel. We first prove them when $\dim(H) = d+1$ by applying the identities developed in Section 3. We then notice two phenomena of dimensionality reduction. The first is that projection reduces the values of $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$. The second is that if $u \in (\mathrm{Sp}(\{v_1, \ldots, v_{d+1}\}))^{\perp}$, then the corresponding simplex inequality for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ reduces to a relaxed simplex inequality of one term and constant 1. We remark that the second phenomenon of dimensionality reduction is not fully necessary for concluding the theorems, i.e., using the regular simplex inequality is fine, but we find it worth mentioning.

We prove Theorem 4.1 in Subsection 4.1 and Theorem 4.2 in Subsection 4.2.

### 4.1. The Proof of Theorem 4.1.

#### 4.1.1. *The Case of* $\mathbf{dim(H) = d+1}$.

We establish the following proposition.

**Lemma 4.1.** *If $\dim(H) = d+1$, $\{v_1, \ldots, v_{d+1}\} \subseteq H$ and $u \in H \setminus \{0\}$, then*

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|. \tag{37}$$

*Proof of Lemma 4.1.* Let $S = \{v_1, \ldots, v_{d+1}\}$. If $S$ is linearly dependent, then $|\operatorname{g}_d\!\sin_0(v_1, \ldots, v_{d+1})| = 0$, and the inequality holds. Similarly, if $u$ is scalar multiple of any of the individual basis vectors $v_1, \ldots, v_{d+1}$, then the inequality holds as an equality. Thus, we may assume that $\mathrm{Sp}(S) = H$ and that $u$ is not a scalar multiple of any of the individual basis vectors $v_1, \ldots, v_{d+1}$.

Furthermore, we may assume that $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$. Indeed, if this is not the case, then we may apply the following procedure. We express $u$ as a linear combination of the vectors $\{v_i\}_{i=1}^{d+1}$ using the coefficients $\{\lambda_i\}_{i=1}^{d+1}$:

$$u = \sum_{i=1}^{d+1} \lambda_i\, v_i = \sum_{i=1}^{d+1} |\lambda_i|\, \operatorname{sign}(\lambda_i)\, v_i, \text{ where } \sum_{i=1}^{d+1} |\lambda_i| \neq 0.$$

For all $1 \leq i \leq d+1$, we let

$$\hat{v}_i = \begin{cases} \operatorname{sign}(\lambda_i)\, v_i, & \text{if } \lambda_i \neq 0, \\ v_i, & \text{otherwise.} \end{cases}$$

We note that $u = \sum_{i=1}^{d+1} |\lambda_i| \cdot \hat{v}_i$, and therefore $u \in \mathrm{C}_{\mathrm{poly}}(\hat{v}_1, \ldots, \hat{v}_{d+1})$. Moreover, by the scale-invariance of the function $|\operatorname{g}_d\!\sin_0|$ we obtain that the required inequality (equation (37)) holds if and only if

$$|\operatorname{g}_d\!\sin_0(\hat{v}_1, \ldots, \hat{v}_{d+1})| \leq \sum_{i=1}^{d+1} |\operatorname{g}_d\!\sin_0(\hat{v}_1, \ldots, \hat{v}_{i-1}, u, \hat{v}_{i+1}, \ldots, \hat{v}_{d+1})|.$$

Thus it is sufficient to consider the case where $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$. We observe that this assumption and equation (15) imply that

$$\tilde{u} \in \mathrm{C}_{\mathrm{hull}}(v_1, \ldots, v_{d+1})\,. \tag{38}$$

We next obtain the desired inequality by using equation (26) together with the form of $\{Q_i\}_{i=1}^{d+1}$ set in equation (28). The question is how to choose the positive coefficients $\{\beta_i\}_{i=1}^{d+1}$ such that $Q_i \leq 1$, $i = 1, \ldots, d+1$. Avoiding a messy optimization argument, we will show that there is a natural geometric choice for the parameters $\{\beta_i\}_{i=1}^{d+1}$. Indeed, letting

$$\beta_i = M_d(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots v_{d+1}),\ i = 1, \ldots, d+1,$$

we have that for all $1 \leq i \leq d+1$

$$\begin{aligned} &M_d(\beta_1 v_1, \ldots \beta_{i-1} v_{i-1}, \beta_{i+1} v_{i+1}, \ldots, \beta_{d+1} v_{d+1}) = \\ &\qquad M_d(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1}) \cdot \prod_{\substack{j=1 \\ j \neq i}}^{d+1} \beta_j = \prod_{j=1}^{d+1} \beta_j = \prod_{j=1}^{d+1} M_d(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{d+1}). \end{aligned} \tag{39}$$

In particular, for the simplex with vertices $\{0, \beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}\}$, we obtain equal contents for all $d$-faces containing the vertex 0.

Another geometric property of the resulting simplex is that if $1 \leq k \neq i \leq d+1$, then $\beta_k v_k$ and $\beta_i v_i$ are of equal distance from the $(d-1)$-plane $L_{S \setminus \{v_i, v_k\}}$. That is,

$$\operatorname{dist}(\beta_k v_k, L_{S \setminus \{v_i, v_k\}}) = \operatorname{dist}(\beta_i v_i, L_{S \setminus \{v_i, v_k\}}), \text{ where } 1 \leq k \neq i \leq d+1.$$

This is a direct result of equation (12) and the fact that the $d$-dimensional contents of the relevant faces are equal (recall equation (39)). Then, denoting the common distance for both $\beta_k v_k$ and $\beta_i v_i$ from $L_{S \setminus \{v_i, v_k\}}$ by $d_{ik}$, we note that

$$\{\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}\} \subseteq \mathrm{T}_{\mathrm{ube}}\left(L_{S \setminus \{v_i, v_k\}}, d_{ik}\right) \quad \text{for all} \quad 1 \leq k \neq i \leq d+1\,.$$

Since $\mathrm{T}_{\mathrm{ube}}\left(L_{S \setminus \{v_i, v_k\}}, d_{ik}\right)$ is convex,

$$\mathrm{C}_{\mathrm{hull}}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) \subseteq \mathrm{T}_{\mathrm{ube}}\left(L_{S \setminus \{v_i, v_k\}}, d_{ik}\right) \quad \text{for all} \quad 1 \leq k \neq i \leq d+1\,.$$

This observation together with equation (38) imply that

$$\tilde{u} \in \mathrm{T}_{\mathrm{ube}}\left(L_{S \setminus \{v_i, v_k\}}, d_{ik}\right) \quad \text{for all} \quad 1 \leq k \neq i \leq d+1,$$

that is,

$$\frac{\operatorname{dist}\left(\tilde{u}, L_{S\setminus\{v_i,v_k\}}\right)}{\operatorname{dist}\left(\beta_i v_i, L_{S\setminus\{v_i,v_k\}}\right)} \leq 1 \quad \text{for all } 1 \leq k \neq i \leq d+1. \tag{40}$$

It follows from equations (28) and (40) that $0 \leq Q_i \leq 1$ for all $1 \leq i \leq d+1$ and the desired inequality is concluded. □

4.1.2. ***Dimensionality Reduction I***. We show that projections reduce the value of $|\operatorname{g}_d\!\sin_0|$.

**Lemma 4.2.** *If $V$ is a $(d+1)$-dimensional subspace of $H$, $\{v_1,\dots,v_d\} \subseteq V$, $u \in H$, and $P_V : H \to V$ is the orthogonal projection onto $V$, then*

$$|\operatorname{g}_d\!\sin_0(v_1,\dots,v_d,P_V(u))| \leq |\operatorname{g}_d\!\sin_0(v_1,\dots,v_d,u)|. \tag{41}$$

*Proof of Lemma 4.2.* We form the sets $B = \{v_1,\dots,v_d\}$, $S = \{v_1,\dots,v_d,u\}$ and $\tilde{S} = \{v_1,\dots,v_d,P_V(u)\}$. In order to conclude the lemma it is sufficient to prove the following inequality for dihedral angles:

$$\sin\left(\alpha\left(L_{\tilde{S}\setminus\{P_V(u)\}}, L_{\tilde{S}\setminus\{v_i\}}\right)\right) \leq \sin\left(\alpha\left(L_{S\setminus\{u\}}, L_{S\setminus\{v_i\}}\right)\right), \quad \text{for all } 1 \leq i \leq d. \tag{42}$$

Indeed, equation (41) is a direct consequence of both equation (42) and the product formula for $|\operatorname{g}_d\!\sin_0|$ of Proposition 2.4.

In order to prove the bound of equation (42) it will be convenient to use the following orthogonal projections, while recalling that $B = \{v_1,\dots,v_d\}$:

$$\begin{aligned} P_B &: H \to L_B, \\ N_B &: H \to (L_B)^{\perp} \cap V, \\ P_i &: H \to L_{B\setminus\{v_i\}},\ 1 \leq i \leq d, \\ N_i &: H \to (L_{B\setminus\{v_i\}})^{\perp} \cap L_B,\ 1 \leq i \leq d. \end{aligned}$$

We also define

$$N_V := I - P_V.$$

We note that $u = P_V(u) + N_V(u) = P_i(u) + N_i(u) + N_B(u) + N_V(u)$, for all $1 \leq i \leq d$.

If $N_B(u) = 0$, then $P_V(u) = P_B(u)$ and the set $\{v_1,\dots,v_d,P_V(u)\}$ is linearly dependent. Hence, $|\operatorname{g}_d\!\sin_0(v_1,\dots,v_d,P_V(u))| = 0$ and the inequality holds in this case.

If $N_B(u) \neq 0$, we apply equation (11) and obtain that

$$\sin\left(\alpha\left(L_{\tilde{S}\setminus\{P_V(u)\}}, L_{\tilde{S}\setminus\{v_i\}}\right)\right) = \frac{\operatorname{dist}\left(P_V(u), L_{\tilde{S}\setminus\{P_V(u)\}}\right)}{\operatorname{dist}\left(P_V(u), L_{\tilde{S}\setminus\{P_V(u),v_i\}}\right)} = \frac{\|N_B(u)\|}{\|N_B(u)+N_i(u)\|}, \quad 1 \leq i \leq d, \tag{43}$$

and

$$\sin\left(\alpha\left(L_{S\setminus\{u\}}, L_{S\setminus\{v_i\}}\right)\right) = \frac{\operatorname{dist}\left(u, L_{S\setminus\{u\}}\right)}{\operatorname{dist}\left(u, L_{S\setminus\{u,v_i\}}\right)} = \frac{||N_B(u)+N_V(u)||}{||N_i(u)+N_B(u)+N_V(u)||}, \quad 1 \leq i \leq d. \tag{44}$$

For any fixed $1 \leq i \leq d$, the vectors $N_B(u)$, $N_V(u)$, and $N_i(u)$ are mutually orthogonal, and therefore

$$\sin\left(\alpha\left(L_{S\setminus\{u\}}, L_{S\setminus\{v_i\}}\right)\right) = \frac{\|N_B(u)\|}{\|N_i(u)+N_B(u)\|}\sqrt{\frac{\frac{||N_V(u)||^2}{||N_B(u)||^2}+1}{\frac{||N_V(u)||^2}{||N_B(u)||^2+||N_i(u)||^2}+1}} \geq \frac{||N_B(u)||}{||N_i(u)+N_B(u)||}. \tag{45}$$

Equation (42) follows from equations (43) and (45), and thus the lemma is concluded. □

4.1.3. ***Dimensionality Reduction II***. We show how to relax the simplex inequality stated in Theorem 4.1 in the following special case.

**Lemma 4.3.** *If $V$ is a $(d+1)$-dimensional subspace of $H$, $\{v_1,\dots,v_{d+1}\} \subseteq V$, and $u \in V^{\perp} \setminus \{0\}$, then*

$$|\operatorname{g}_d\!\sin_0(v_1,\dots v_{d+1})| \leq |\operatorname{g}_d\!\sin_0(v_1,\dots,v_{i-1},u,v_{i+1},\dots,v_{d+1})|, \quad \text{for all } 1 \leq i \leq d+1. \tag{46}$$

*Proof of Lemma 4.3.* We assume without loss of generality that $\mathrm{Sp}(\{v_1,\dots,v_{d+1}\}) = V$ (otherwise equation (46) follows trivially). We define the sets $S_i = \{v_1,\dots,v_{i-1},u,v_{i+1},\dots,v_{d+1}\}$ for all $1 \le i \le d+1$. Since $u \in V^\perp \setminus \{0\}$ we obtain from equation (11) that

$$\sin\left(\alpha\left(L_{S_i\setminus\{u\}}, L_{S_i\setminus\{v_j\}}\right)\right) = 1, \quad \text{for all } 1 \le j \ne i \le d+1. \tag{47}$$

Combining equation (47) with the product formula for $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ (Proposition 2.4) we get the following equality for all $1 \le i \le d+1$,

$$\begin{aligned} &|\,\mathrm{g}_d\mathrm{sin}_0(v_1,\dots,v_{i-1},u,v_{i+1},\dots,v_{d+1})|^d \\ &\quad = |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1,\dots,v_{i-1},v_{i+1},\dots,v_{d+1})|^{d-1} \prod_{\substack{j=1\\ j\ne i}}^{d+1} \sin\left(\alpha\left(L_{S_i\setminus\{u\}}, L_{S_i\setminus\{v_j\}}\right)\right) \\ &\quad = |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1,\dots,v_{i-1},v_{i+1},\dots,v_{d+1})|^{d-1}. \end{aligned} \tag{48}$$

By further application of the product formula for $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ we obtain that for all $1 \le i \le d+1$,

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1,\dots v_{d+1})|^d \le |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1,\dots,v_{i-1},v_{i+1}\dots,v_{d+1})|^{d-1}. \tag{49}$$

Equation (46) thus follows from equations (48) and (49). ☐

4.1.4. ***Conclusion of Theorem 4.1***. Let $P$ denote the orthogonal projection from $H$ onto $\mathrm{Sp}\{v_1,\dots,v_{d+1}\}$. If $P(u) = 0$, then we conclude the Theorem from Lemma 4.3.

If $P(u) \ne 0$, then we conclude the theorem by applying Lemmata 4.1 and 4.2 successively as follows:

$$\begin{aligned} |\,\mathrm{g}_d\mathrm{sin}_0(v_1,\dots,v_{d+1})| &\le \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(v_1,\dots,v_{i-1},P(u),v_{i+1},\dots,v_{d+1})| \\ &\le \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(v_1,\dots v_{i-1},u,v_{i+1}\dots,v_{d+1})|. \quad \square \end{aligned}$$

4.2. **The Proof of Theorem 4.2.** Here we prove essentially the same three lemmata of Subsection 4.1 for the function $|\,\mathrm{p}_d\mathrm{sin}_0\,|$.

4.2.1. ***The Case of*** $\mathbf{dim(H) = d+1}$. We establish here the following proposition.

**Lemma 4.4.** *If* $\dim(H) = d+1$, $v_1,\dots,v_{d+1} \in H$ *and* $u \in H \setminus \{0\}$, *then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1,\dots,v_{d+1})| \le \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1,\dots,v_{i-1},u,v_{i+1},\dots,v_{d+1})|. \tag{50}$$

*Proof of Lemma 4.4.* Similarly as in the proof of Lemma 4.1, we can assume that $v_1,\dots,v_{d+1}$ are linearly independent, $u$ is not a scalar multiple of any of the individual basis vectors $v_1,\dots,v_{d+1}$ and $u \in \mathrm{C_{poly}}(v_1,\dots,v_{d+1})$. Using the choice of $\{\beta_i\}_{i=1}^{d+1}$ specified in equation (20), we have that $\|\beta_i v_i\| \le 1$ for all $1 \le i \le d+1$. In view of equation (15) we can extend this bound to $\tilde{u}$, i.e., we have that $\|\tilde{u}\| \le 1$. The lemma then follows by combining equation (22) with the latter bound. ☐

4.2.2. ***Dimensionality Reduction I***. We show that projections reduce the value of $|\,\mathrm{p}_d\mathrm{sin}_0\,|$.

**Lemma 4.5.** *If* $V$ *is a* $(d+1)$*-dimensional subspace of* $H$, $v_1,\dots,v_d \in V$, $u \in H$, *and* $P_V : H \to V$ *is the orthogonal projection onto* $V$, *then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1,\dots,v_d,P_V(u))| \le |\,\mathrm{p}_d\mathrm{sin}_0(v_1,\dots,v_d,u)|. \tag{51}$$

*Proof of Lemma 4.5.* We form the sets $B = \{v_1,\dots v_d\} \subseteq V$, $S = \{v_1,\dots,v_d,u\}$ and $\tilde{S} = \{v_1,\dots,v_d,P_V(u)\}$. In order to conclude the lemma, it is sufficient to prove that

$$\sin\left(\theta\left(P_V(u), L_{\tilde{S}\setminus\{P_V(u)\}}\right)\right) \le \sin\left(\theta\left(u, L_{S\setminus\{u\}}\right)\right). \tag{52}$$

Indeed, equation (51) is a direct consequence of equation (52) and the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ (Proposition 2.5).

In order to prove equation (52), it will be convenient to use the following orthogonal projections:

$$P_B : H \to L_B,$$
$$N_B : H \to (L_B)^{\perp} \cap V.$$

We also define

$$N_V := I - P_V \,.$$

We note that $u = P_V(u) + N_V(u) = P_B(u) + N_B(u) + N_V(u)$.

If $N_B(u) = 0$, then $P_V(u) = P_B(u) \in L_B$, and the inequality (equation (51)) holds trivially since the set $\tilde{S} = \{v_1, \ldots, v_d, P_V(u)\}$ is linearly dependent.

If $N_B(u) \neq 0$, we apply equation (9) to obtain that

$$\sin\left(\theta\left(u, L_{S\setminus\{u\}}\right)\right) = \frac{\operatorname{dist}\left(u, L_{S\setminus\{u\}}\right)}{\|u\|} = \frac{\|N_B(u) + N_V(u)\|}{\|P_B(u) + N_B(u) + N_V(u)\|}\,,$$

and

$$\sin\left(\theta\left(P_V(u), L_{\tilde{S}\setminus\{P_V(u)\}}\right)\right) = \frac{\operatorname{dist}\left(P_V(u), L_{\tilde{S}\setminus\{P_V(u)\}}\right)}{\|P_V(u)\|} = \frac{\|N_B(u)\|}{\|P_B(u) + N_B(u)\|}\,.$$

Thus,

$$\sin\left(\theta\left(u, L_{S\setminus\{u\}}\right)\right) = \sin\left(\theta\left(P_V(u), L_{\tilde{S}\setminus\{P_V(u)\}}\right)\right)\sqrt{\frac{1 + \frac{\|N_V(u)\|^2}{\|N_B(u)\|^2}}{1 + \frac{\|N_V(u)\|^2}{\|P_B(u)+N_B(u)\|^2}}} \geq \sin\left(\theta\left(P_V(u), L_{\tilde{S}\setminus\{P_V(u)\}}\right)\right).$$

That is, equation (52) is verified and the lemma is concluded. □

4.2.3. ***Dimensionality Reduction II***. We show how to relax the simplex inequality stated in Theorem 4.2 in the following special case.

**Lemma 4.6.** *If $V$ is a $(d+1)$-dimensional subspace of $H$, $v_1, \ldots, v_{d+1} \in V$, and $u \in V^{\perp} \setminus \{0\}$, then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|, \quad \text{for } i = 1, \ldots, d+1\,. \tag{53}$$

*Proof of Lemma 4.6.* We assume without loss of generality that $\mathrm{Sp}(\{v_1, \ldots, v_{d+1}\}) = V$. We define the sets $S_i = \{v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}\}$ for all $1 \leq i \leq d+1$. Since $u \in V^{\perp} \setminus \{0\}$ we obtain from equation (9) that

$$\sin\left(\theta\left(u, L_{S_i\setminus\{u\}}\right)\right) = 1, \quad \text{for all } 1 \leq i \leq d+1\,. \tag{54}$$

Combining equation (54) with the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ (Proposition 2.5), we get the following equality for all $i = 1, \ldots, d+1$

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})| = |\,\mathrm{p}_{d\text{-}1}\mathrm{sin}_0\,(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1})\,|\,. \tag{55}$$

By further application of the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$, we obtain that for all $i = 1, \ldots, d+1$:

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq |\,\mathrm{p}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1})|. \tag{56}$$

Combining equations (55) and (56) we conclude equation (53). □

4.2.4. ***Conclusion of Theorem 4.2***. Let $P$ denote the orthogonal projection of $H$ onto $\mathrm{Sp}(\{v_1, \ldots, v_{d+1}\})$. If $P(u) = 0$, then we conclude the theorem from Lemma 4.6.

If $P(u) \neq 0$, then applying Lemmata 4.4 and 4.5 successively we obtain that

$$\begin{aligned}|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| &\leq \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, P(u), v_{i+1}, \ldots, v_{d+1})| \\ &\leq \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots v_{i-1}, u, v_{i+1} \ldots, v_{d+1})|\,. \quad \square\end{aligned}$$

## 5. Ahlfors Regular Measures and Concentration Inequalities for the Polar Sine

In this section we prove Theorem 1.2. As explained in the introduction, we interpret this theorem as indicating that the polar sine $|\,\mathrm{p}_d\mathrm{sin}_w\,|$ satisfies a relaxed simplex inequality of two terms with "high Ahlfors probability at all scales and locations". Both scales and locations are given by balls centered on $\mathrm{supp}(\mu)$, and probabilities are given by scaling the $\gamma$-dimensional Ahlfors regular measures of such balls, where $d-1<\gamma\leq d$.

5.1. **Notation, Definitions and Elementary Propositions.** For convenience of our notation, we assume that $w=0$ and $0\in\mathrm{supp}(\mu)$, and thus establish most of the propositions for $\mathrm{p}_d\mathrm{sin}_0$. They can be generalized for $\mathrm{p}_d\mathrm{sin}_w$ via equation (8).

Throughout this section we extensively use the definitions and notation for elevation, maximal elevation, and dihedral angels formulated in Subsection 2.5. We often fix $S=\{v_1,\dots,v_{d+1}\}\subseteq H$. If $0\leq\epsilon\leq 1$ and $1\leq i\leq d+1$, then we denote by ${\mathrm{C_{one}}}^i(\epsilon)$, the cone

$$\mathrm{C_{one}}^i(\epsilon)=\mathrm{C_{one}}\left(\epsilon\cdot\theta\left(v_i,L_{S\setminus\{v_i\}}\right),L_{S\setminus\{v_i\}},0\right).$$

If $0\leq\epsilon\leq 1$ and $1\leq i<j\leq d+1$, then we denote by $\mathrm{C_{one}}^{i,j}(\epsilon)$ the set

$$\mathrm{C_{one}}^{i,j}(\epsilon)=\mathrm{C_{one}}^i(\epsilon)\cap\mathrm{C_{one}}^j(\epsilon)\,. \tag{57}$$

If $1\leq i<j\leq d+1$, then we denote by $\Theta_{i,j}$ the following maximal elevation angle

$$\Theta_{i,j}=\Theta(v_i,v_j,L_{S\setminus\{v_i,v_j\}}). \tag{58}$$

Throughout the rest of the paper we fix a real parameter $\gamma\in\mathbb{R},\ d-1<\gamma\leq d$ (the most natural choice is $\gamma=d$) and assume that $H$ is equipped with a $\gamma$-dimensional Ahlfors regular measure, which we define as follows:

**Definition 5.1.** *A locally finite Borel measure $\mu$ on $H$ is a $\gamma$-dimensional Ahlfors regular measure if there exists a constant $C$ such that for all $x\in\mathrm{supp}(\mu)$ and $0<r\leq\mathrm{diam}(\mathrm{supp}(\mu))$,*

$$C^{-1}\cdot r^{\gamma}\leq\mu(B(x,r))\leq C\cdot r^{\gamma}.$$

We denote the smallest constant $C$ for which the inequality above holds by $C_\mu$. We refer to it as the *regularity constant* of $\mu$.

The following proposition and its immediate corollary, will be useful for us. We prove them in Appendix A.3.

**Proposition 5.1.** *If $\gamma>1$, $m\in\mathbb{N}$ such that $1\leq m<\gamma$, $\mu$ a $\gamma$-dimensional Ahlfors regular measure on $H$ with regularity constant $C_\mu$, $0\leq\epsilon\leq 1$, and $L$ an $m$-dimensional affine subspace of $H$, then for all $x\in\mathrm{supp}(\mu)\cap L$ and $0<r\leq\mathrm{diam}(\mathrm{supp}(\mu))$*

$$\mu(\mathrm{T_{ube}}(L,\epsilon\cdot r)\cap B(x,r))\ \leq\ 2^{m+\frac{3\gamma}{2}}\cdot C_\mu\cdot\epsilon^{\gamma-m}\cdot r^{\gamma}. \tag{59}$$

**Corollary 5.1.** *If $\gamma>1$, $m\in\mathbb{N}$ such that $1\leq m<\gamma$, $\mu$ a $\gamma$-dimensional Ahlfors regular measure on $H$ with regularity constant $C_\mu$, $0\leq\theta\leq\pi/2$, and $L$ an $m$-dimensional affine subspace of $H$, then for all $x\in\mathrm{supp}(\mu)\cap L$ and $0<r\leq\mathrm{diam}(\mathrm{supp}(\mu))$*

$$\mu(\mathrm{C_{one}}(\theta,L,x)\cap B(x,r))\ \leq\ 2^{m+\frac{3\gamma}{2}}\cdot C_\mu\cdot\sin(\theta)^{\gamma-m}\cdot r^{\gamma}\,.$$

We will frequently use the following elementary inequalities for the one-dimensional sine:

**Lemma 5.1.** *If $0\leq\theta\leq\frac{\pi}{2}$ and $0\leq c\leq 1$, then*

$$c\cdot\sin(\theta)\leq\sin(c\cdot\theta) \tag{60}$$

*and*

$$\sin(c\cdot\theta)\leq\frac{\pi}{2}\cdot c\cdot\sin(\theta). \tag{61}$$

Both inequalities can be derived by noting that the function $\sin(c\,\theta)/(c\,\sin(\theta))$ is increasing in $\theta$ and thus obtains its lower bound, 1, as $\theta$ approaches 0 and its maximum value, bounded by $\pi/2$, at $\theta=\frac{\pi}{2}$.

5.2. **Relationship between Conic Regions and Relaxed Inequalities for the Polar Sine.** We establish here the following relation between the set $U_C(S,0)$ defined in equation (4) and the intersection of various cones.

**Proposition 5.2.** *If $S = \{v_1, \ldots, v_{d+1}\} \subseteq H$, $C \geq 1$, $U_C(S,0)$ is the set defined in equation (4) with $w = 0$, and ${\mathrm{C_{one}}}^{i,j}(C^{-1})$ for $1 \leq i < j \leq d+1$ are the intersections of cones defined in equation (57) with $\epsilon = C^{-1}$, then*

$$H \setminus \left( \bigcup_{1 \leq i < j \leq d+1} {\mathrm{C_{one}}}^{i,j}(C^{-1}) \right) \subseteq U_C(S,0).$$

*Proof.* We note that

$$H \setminus \bigcup_{1 \leq i < j \leq d+1} {\mathrm{C_{one}}}^{i,j}(C^{-1}) = \bigcap_{1 \leq i < j \leq d+1} \left( H \setminus {\mathrm{C_{one}}}^{i,j}(C^{-1}) \right),$$

and

$$U_C(S,0) = \bigcap_{1 \leq i < j \leq d+1} U_C^{i,j}(S,0)\,,$$

where

$$U_C^{i,j}(S,0) = \Big\{ u \in H : |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq C \cdot \Big( |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})| + |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_{d+1})| \Big) \Big\}. \tag{62}$$

Therefore, in order to conclude the proposition we only need to prove that

$$H \setminus {\mathrm{C_{one}}}^{i,j}(C^{-1}) \subseteq U_C^{i,j}(S,0), \quad \text{for all } 1 \leq i < j \leq d+1. \tag{63}$$

If $u \in H \setminus {\mathrm{C_{one}}}^{i,j}(C^{-1})$ for some $i$ and $j$, where $1 \leq i < j \leq d+1$, then either $u \in H \setminus {\mathrm{C_{one}}}^{i}(C^{-1})$ or $u \in H \setminus {\mathrm{C_{one}}}^{j}(C^{-1})$. Assume without loss of generality that $u \in H \setminus {\mathrm{C_{one}}}^{i}(C^{-1})$, then

$$\sin\left(\theta\left(u, L_{S \setminus \{v_i\}}\right)\right) \geq \sin\left(C^{-1} \cdot \theta\left(v_i, L_{S \setminus \{v_i\}}\right)\right). \tag{64}$$

Combining the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0|$ (Proposition 2.5) with equations (64) and (60), we obtain that

$$C \cdot |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})| \geq |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})|.$$

In particular, $u \in U_C^{i,j}(S,0)$, and equation (63), and consequently the proposition, is concluded. $\square$

5.3. **Controlling the Intersection of Two Cones.** The main part of the proof of Theorem 1.2 is to show that for any $0 \leq \epsilon \leq 1$ we can control the measure of the sets ${\mathrm{C_{one}}}^{i,j}(\epsilon)$ which are defined in equation (57). We accomplish this by showing that such sets are contained in specific cones on $(d-1)$-dimensional subspaces of $H$, and then applying Corollary 5.1 to control the measure of the latter cones. The crucial proposition is the following.

**Proposition 5.3.** *If $0 \leq s \leq 1$, $k \geq 3$, $V$ is a $k$-dimensional subspace of $H$, $V_1$ and $V_2$ are two different $(k-1)$-dimensional subspaces of $V$, $v_1 \in V_1 \setminus V_2$, $v_2 \in V_2 \setminus V_1$, and $\theta(v_1, V_2)$, $\theta(v_2, V_1)$, as well as $\Theta(v_1, v_2, V_1 \cap V_2)$ are the corresponding elevation and maximal elevation angles, then*

$$\mathrm{C_{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi} \cdot \theta\left(v_1, V_2\right), V_2, 0\right) \bigcap \mathrm{C_{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi} \cdot \theta\left(v_2, V_1\right), V_1, 0\right) \subseteq \mathrm{C_{one}}\left(s \cdot \Theta(v_1, v_2, V_1 \cap V_2), V_1 \cap V_2, 0\right).$$

*Proof.* Note that $\dim\left(V_1 \cap V_2\right) = k - 2$, and that

$$V = \mathrm{Sp}\{v_1\} + V_2 = \mathrm{Sp}\{v_2\} + V_1 = \mathrm{Sp}\{v_1, v_2\} + V_1 \cap V_2.$$

However, the above sum is not direct (the subspaces in the sum are not mutually orthogonal). We thus create a few orthogonal subspaces which are expressed via the following orthogonal projections:

$$\begin{aligned} P_i &: H \to V_i, \ i = 1, 2, \\ N_i &: H \to V_i \cap (V_1 \cap V_2)^{\perp}, \ i = 1, 2, \\ P_{1,2} &: H \to V_1 \cap V_2. \end{aligned}$$

Note that

$$P_i = P_{1,2} + N_i, \quad i = 1, 2, \tag{65}$$

and consequently

$$(I - P_2) \cdot P_1 = (I - P_2) \cdot N_1. \tag{66}$$

We denote

$$\begin{aligned}
\tilde{\mathrm{C}}^1_{\mathrm{one}} &= \mathrm{C}_{\mathrm{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi}\cdot\theta\left(v_2, V_1\right), V_1, 0\right),\\
\tilde{\mathrm{C}}^2_{\mathrm{one}} &= \mathrm{C}_{\mathrm{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi}\cdot\theta\left(v_1, V_2\right), V_2, 0\right),\\
\tilde{\mathrm{C}}^{1,2}_{\mathrm{one}} &= \tilde{\mathrm{C}}^1_{\mathrm{one}} \cap \tilde{\mathrm{C}}^2_{\mathrm{one}},\\
\tilde{\Theta}_{1,2} &= \Theta(v_1, v_2, V_1 \cap V_2).
\end{aligned}$$

Following our notation and the definition of a cone, we need to prove that

$$\|u - P_{1,2}(u)\| \le \sin\left(s \cdot \tilde{\Theta}_{1,2}\right) \cdot \|u\|, \quad \text{for all } u \in \tilde{\mathrm{C}}^{1,2}_{\mathrm{one}},$$

or equivalently (via equation (65)),

$$\|N_1(u) + u - P_1(u)\| \le \sin\left(s \cdot \tilde{\Theta}_{1,2}\right) \cdot \|u\|, \quad \text{for all } u \in \tilde{\mathrm{C}}^{1,2}_{\mathrm{one}}. \tag{67}$$

For $u \in \tilde{\mathrm{C}}^{1,2}_{\mathrm{one}}$, we will bound $\|N_1(u)\|$ and $\|u - P_1(u)\|$ separately and then combine the two estimates to conclude the above inequality and the current proposition.

Our bound for $\|u - P_1(u)\|$ is straightforward. Indeed, if $u \in \tilde{\mathrm{C}}^{1,2}_{\mathrm{one}}$ respectively, then $u \in \tilde{\mathrm{C}}^1_{\mathrm{one}}$, and by the definition of $\tilde{\mathrm{C}}^1_{\mathrm{one}}$ as well as the application of equation (61) we obtain that

$$\|u - P_1(u)\| \le \sin\left(\frac{2\,s}{\sqrt{5}\,\pi}\cdot\theta(v_2, V_1)\right)\cdot\|u\| \le \frac{s}{\sqrt{5}}\cdot\sin\left(\theta(v_2, V_1)\right)\cdot\|u\|. \tag{68}$$

Our bound for $\|N_1(u)\|$ has the following form:

$$\|N_1(u)\| \le \frac{s}{\sqrt{5}}\left[\sin(\theta(v_1, V_1 \cap V_2)) + \sin(\theta(v_2, V_1 \cap V_2))\right]\cdot\|u\|. \tag{69}$$

In order to verify it, we assume without loss of generality that $N_1(u) \neq 0$ and note that equation (11) implies the following relation

$$\sin(\alpha(V_1, V_2)) = \frac{\mathrm{dist}(N_1(u), V_2)}{\mathrm{dist}(N_1(u), V_1 \cap V_2)} = \frac{\|N_1(u) - P_2 \cdot N_1(u)\|}{\|N_1(u)\|}. \tag{70}$$

Combining equations (66) and (70) we obtain that

$$\|N_1(u)\| = \frac{\|P_1(u) - P_2 \cdot P_1(u)\|}{\sin\left(\alpha(V_1, V_2)\right)}. \tag{71}$$

We bound $\|P_1(u) - P_2 \cdot P_1(u)\|$ as follows.

$$\begin{aligned}
\|P_1(u) - P_2 \cdot P_1(u)\| &= \|(I - P_2)\cdot P_1(u)\| \\
&= \|(I - P_2)(u) - (I - P_2)\cdot(I - P_1)(u)\| \le \|u - P_2(u)\| + \|u - P_1(u)\|.
\end{aligned} \tag{72}$$

Equation (68) gives a bound for $\|u - P_1(u)\|$, and similarly we obtain that

$$\|u - P_2(u)\| \le \frac{s}{\sqrt{5}}\cdot\sin(\theta(v_1, V_2)\cdot\|u\|. \tag{73}$$

Combining equations (68) and (71)-(73) we get that

$$\|N_1(u)\| \le \frac{s}{\sqrt{5}}\left(\frac{\sin\left(\theta\left(v_1, V_2\right)\right)}{\sin(\alpha(V_1, V_2))} + \frac{\sin\left(\theta\left(v_2, V_1\right)\right)}{\sin(\alpha(V_1, V_2))}\right)\cdot\|u\|. \tag{74}$$

At last we note that equations (9) and (11) imply that

$$\sin(\alpha(V_1, V_2)) = \frac{\sin(\theta(v_1, V_2))}{\sin(\theta(v_1, V_1 \cap V_2))} = \frac{\sin(\theta(v_2, V_1))}{\sin(\theta(v_2, V_1 \cap V_2))}.$$

Applying this identity in (74), we achieve the bound for $\|N_1(u)\|$ stated in equation (69).

Finally, noting that $N_1(u) \perp (u - P_1(u))$ and applying the bounds of equations (68) and (69) we obtain that

$$\|N_1(u) + u - P_1(u)\|^2 = \|N_1(u)\|^2 + \|u - P_1(u)\|^2$$
$$\leq \left(\frac{s}{\sqrt{5}}\right)^2 \cdot 5 \cdot \sin^2\left(\tilde{\Theta}_{1,2}\right) \cdot \|u\|^2 = s^2 \cdot \sin^2\left(\tilde{\Theta}_{1,2}\right) \cdot \|u\|^2 \leq \sin^2\left(s \cdot \tilde{\Theta}_{1,2}\right) \cdot \|u\|^2.$$

Equation (67), and consequently the proposition, is thus concluded. □

**Remark 5.1.** *The proposition extends trivially to $k = 2$, where the intersection of two cones, centered around two vectors $w_1$ and $w_2$ respectively with opening angles less than half the angle between, is the origin, which is a degenerate cone.*

Proposition 5.3 implies the following corollary:

**Corollary 5.2.** *If $0 \leq s \leq 1$, $2 \leq k \leq \dim(H)$, $1 \leq i < j \leq d+1$, $S = \{v_1, \ldots, v_k\} \subseteq H$ is a linearly independent set and $\Theta_{i,j}$ as well as ${\mathrm{C}_{\mathrm{one}}}^{i,j}\left(\frac{2\cdot s}{\sqrt{5}\,\pi}\right)$ are defined by equations (58) and (57) respectively, then*

$$(75)\qquad {\mathrm{C}_{\mathrm{one}}}^{i,j}\left(\frac{2\cdot s}{\sqrt{5}\,\pi}\right) \subseteq \mathrm{C}_{\mathrm{one}}\left(s \cdot \Theta_{i,j}, L_{S\setminus\{v_i,v_j\}}, 0\right).$$

Indeed, Corollary 5.2 is obtained as a special case of Proposition 5.3 by setting $V = L_S$, $V_1 = L_{S\setminus\{v_i\}}$, and $V_2 = L_{S\setminus\{v_j\}}$, and noting that $V_1 \cap V_2 = L_{S\setminus\{v_i,v_j\}}$.

5.4. **Conclusion of Theorem 1.2.** Theorem 1.2 follows directly from Proposition 5.2 and Corollaries 5.1 and 5.2.

In view of equation (8), we note that it is sufficient to prove the theorem when $w = 0$ and $0 \in \mathrm{supp}(\mu)$. We assume an arbitrary parameter $0 < s \leq 1$ and set

$$(76)\qquad C = \frac{\sqrt{5}\,\pi}{2\cdot s}.$$

At the end of the proof we further restrict the values of $s$ from above and consequently restrict those of $C$ from below.

Let $S = \{v_1, \ldots, v_{d+1}\} \subseteq H$, $0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu))$, and ${\mathrm{C}_{\mathrm{one}}}^{i,j}(C^{-1})$ be defined according to equation (57). We assume without loss of generality that the set $S$ is linearly independent. Proposition 5.2 implies that

$$B(0,r) \setminus \left(\bigcup_{1\leq i\neq j\leq d+1} {\mathrm{C}_{\mathrm{one}}}^{i,j}(C^{-1})\right) \subseteq B(0,r) \bigcap U_C(S,0).$$

Using the additivity and monotonicity of $\mu$, we get

$$(77)\qquad \mu\left(B(0,r) \cap U_C(S,0)\right) \geq \mu\left(B(0,r)\right) \; - \sum_{1\leq i<j\leq d+1} \mu\left(B(0,r) \cap {\mathrm{C}_{\mathrm{one}}}^{i,j}(C^{-1})\right).$$

Next, we combine Corollary 5.2 together with equation (76) to obtain that

$$(78)\quad B(0,r) \cap {\mathrm{C}_{\mathrm{one}}}^{i,j}(C^{-1}) \subseteq B(0,r) \cap \mathrm{C}_{\mathrm{one}}\left(s \cdot \Theta_{i,j}, L_{S\setminus\{v_i,v_j\}}, 0\right) \subseteq B(0,r) \cap \mathrm{C}_{\mathrm{one}}\left(s \cdot \frac{\pi}{2}, L_{S\setminus\{v_i,v_j\}}, 0\right).$$

Now, Corollary 5.1, Definition 5.1, and equation (78) imply that for all $1 \leq i \neq j \leq d+1$,

$$(79)\quad \mu\left(B(0,r) \cap {\mathrm{C}_{\mathrm{one}}}^{i,j}(C^{-1})\right) \leq \mu\left(B(0,r) \cap \mathrm{C}_{\mathrm{one}}\left(s \cdot \frac{\pi}{2}, L_{S\setminus\{v_i,v_j\}}, 0\right)\right)$$
$$\leq 2^{\frac{3\gamma}{2}+d-1} \cdot C_\mu^2 \cdot \left(\sin\left(s \cdot \frac{\pi}{2}\right)\right)^{\gamma+1-d} \cdot \mu(B(0,r)).$$

Combining equations (77) and (79), we get that

$$(80)\qquad \frac{\mu\big(B(0,r) \cap U_C(S,0)\big)}{\mu\left(B(0,r)\right)} \geq 1 \; - \; \binom{d+1}{2} \cdot 2^{\frac{3\gamma}{2}+d-1} \cdot C_\mu^2 \cdot \left(\sin\left(s \cdot \frac{\pi}{2}\right)\right)^{\gamma+1-d}.$$

By setting the parameter $s$ so that

$$\binom{d+1}{2} \cdot 2^{\frac{3\gamma}{2}+d-1} \cdot C_\mu^2 \cdot \left(\sin\left(s \cdot \frac{\pi}{2}\right)\right)^{\gamma+1-d} \leq \epsilon\,,$$

that is,

$$s \leq s_0' = \frac{2}{\pi} \cdot \arcsin\left[\left(\frac{\epsilon}{2^{\frac{3\gamma}{2}+d-1} \cdot C_\mu^2 \cdot \binom{d+1}{2}}\right)^{\frac{1}{\gamma+1-d}}\right],$$

we obtain that equation (5) is satisfied for all $C \geq C_0'$, where

$$C_0' = \frac{\sqrt{5}\,\pi}{2\,s_0'} = \sqrt{5}\,\left(\frac{\pi}{2}\right)^2 \left(\arcsin\left[\left(\frac{\epsilon}{2^{\frac{3\gamma}{2}+d-1} \cdot C_\mu^2 \cdot \binom{d+1}{2}}\right)^{\frac{1}{\gamma+1-d}}\right]\right)^{-1}$$
$$= O\left(\left(2^{\frac{3\gamma}{2}+d} \cdot C_\mu^2 \cdot \binom{d+1}{2} \cdot \epsilon^{-1}\right)^{\frac{1}{\gamma+1-d}}\right) \quad \text{as } \epsilon \to 0 \;\text{ or }\; d \to \infty\,.$$

The theorem is thus concluded, where $C_0'$ provides an upper bound for the best possible choice for the constant $C_0$. □

**Remark 5.2.** *Note that Theorem 1.2 extends trivially to the case where $\gamma > d$. In fact, in this case, it is possible to replace the set $U_C(S, w)$ by*

$$U_C'(S,w) = \big\{u \in H : \ |\,\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})| \leq$$
$$C \cdot |\,\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|, \text{ for all } 1 \leq i \leq d+1\big\}.$$

*That is, if $\gamma > d$, then the polar sine satisfies a relaxed simplex inequality of one term "with high Ahlfors probability at all scales and locations". This fact is a direct consequence of Corollary 5.1 and analogues of Proposition 5.2 and equation (77) obtained by replacing $U_C(S,0)$ with $U_C'(S,0)$ and $\{{\mathrm{C}_{\mathrm{one}}}^{i,j}(C^{-1})\}_{1\leq i<j\leq d+1}$ with $\{{\mathrm{C}_{\mathrm{one}}}^{i}(C^{-1})\}_{1\leq i<d+1}$.*

*Nevertheless, if $d-1 < \gamma \leq d$, then one cannot replace the set $U_C(S,w)$ in Theorem 1.2 by $U_C'(S,w)$.*

**Remark 5.3.** *Let us slightly reformulate the above results so that they could be directly applied in* [14]. *For $S = \{v_1, \ldots, v_{d+1}\}$ as above, $C > 0$, and an arbitrarily fixed pair of indices $i$ and $j$, where $1 \leq i < j \leq d+1$, we form the set $U_C(S,i,j,0)$ as follows:*

$$U_C(S,i,j,0) = \Big\{u \in H : \ |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq C \cdot \big(|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})| \,+$$
$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_{d+1})|\big)\Big\}.$$

*If $\gamma = d$ and $0 < \epsilon < 1$, then for all $C \geq C_0''$, where*

$$C_0'' = \sqrt{5} \cdot \left(\frac{\pi}{2}\right)^2 \cdot \left(\arcsin\left(\frac{\epsilon}{2^{\frac{5d}{2}-1} \cdot C_\mu^2}\right)\right)^{-1},$$

*we have that*

$$\frac{\mu\big(B(0,r) \cap U_C(S,i,j,0)\big)}{\mu\,(B(0,r))} \geq 1 - \epsilon.$$

## 6. Conclusions and Further Directions

The work presented here touches on both old and modern research. We would like to conclude it by indicating various directions where one can extend it.

**High-Dimensional Menger-type curvature.** In [14, 15] we build on the research presented here to form $d$-dimensional Menger-type curvatures of any integer dimension $d > 1$, and show how they characterize $d$-dimensional uniform rectifiability of $d$-dimensional Ahlfors regular measures on real separable Hilbert spaces.

**The high-dimensional $\mathbf{A} = \mathbf{B}$ Paradigm.** Petkovšek, Wilf, and Zeilberger [19] have presented concrete strategies to prove various identities. However, when considering the high-dimensional sine functions, it is not clear whether a general mechanism exists. The product formulas (Propositions 2.4 and 2.5) simplify the representation of $\mathrm{p}_d\mathrm{sin}_0$ and $\mathrm{g}_d\mathrm{sin}_0$, but they do not seem to provide sufficiently simple structure for automatically proving general identities involving those functions. We have demonstrated additional strategies for proving identities of interest to us and inquire about other useful identities and the strategies for proving them.

**Solutions of high-dimensional functional equations.** We have shown that equation (31) characterizes the generalized sine function of spaces with constant one-dimensional curvature among all Lebesgue measurable functions (Theorem 3.1). It will be interesting to formulate a theorem analogous to Theorem 3.1 for the high-dimensional functional equations described in Section 3. In particular, we are interested in the functional equation generalizing the combination of equations (24) and (27). We could not identify any similar functional equation in the substantial body of work on the subject (see e.g., [1, 2] and references therein).

**Other relaxed inequalities with high probability.** We inquire about probabilistic settings different than the one in here, where the polar sine satisfies a relaxed simplex inequality of two terms, but not of one term, with high probability. We also inquire about other probabilistic settings in which the polar sine satisfies a relaxed simplex inequality of $p$ terms, $3 \leq p \leq d$, and not of $p-1$ terms, with high probability. Moreover, we are curious about probabilistic settings where one can obtain relaxed simplex inequalities for $|\,\mathrm{g}_d\mathrm{sin}\,|$ with high probabilities.

**Applications to data analysis.** Recently, researchers in machine learning have been interested in multi-way clustering and $d$-way kernel methods [3, 21]. Guangliang Chen and the first author [5, 6] have adapted the theory developed here and in [14, 15] to solve a problem of multi-way clustering.

## Appendix A.

A.1. **Proof of Proposition 2.2.** For $\dim(H) > d+1$, the content functions $M_d$ and $M_{d+1}$, and the norm $\|\cdot\|$ are orthogonally invariant and thus $\mathrm{p}_d\mathrm{sin}_0$ and $\mathrm{g}_d\mathrm{sin}_0$ are orthogonally invariant. Moreover, in this case, $M_d$ and $M_{d+1}$ as well as the norm $\|\cdot\|$ scale linearly. That is, for all $1 \leq j \leq d+1$ and $\{\beta_i\}_{i=1}^{d+1}$ such that $\beta_i \neq 0$, where $1 \leq i \leq d+1$:

$$M_d(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1}) = \prod_{i \neq j} |\beta_i| \cdot M_d\left(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{d+1}\right),$$

$$M_{d+1}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \prod_{i=1}^{d+1} |\beta_i| \cdot M_{d+1}(v_1, \ldots, v_{d+1}),$$

and $\|\beta_i v_i\| = |\beta_i| \cdot \|v_i\|$. One can then observe that both the numerator and denominator of $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ scale similarly and thus the latter functions are invariant under nonzero dilations. Similarly, the proposition is satisfied when $\dim(H) = d+1$. $\square$

A.2. **On the positivity of the coefficients $\{\mathbf{Q_i}\}_{\mathbf{i=1}}^{\mathbf{d+1}}$ defined by equation (27).** We show here that the numerators and denominators of the terms $Q_i$, $1 \leq i \leq d+1$, defined by equation (27), have the same signs and thus conclude that these terms are positive.

For $1 \leq i \neq j \leq d+1$ we have that

$$\begin{aligned}
&\mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\tilde{u}}(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, 0, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})] \\
&\qquad = \mathrm{sign}[\det(\beta_1 v_1 - \tilde{u}, \ldots, \beta_{j-1} v_{j-1} - \tilde{u}, -\tilde{u}, \beta_{j+1} v_{j+1} - \tilde{u}, \ldots, \beta_{d+1} v_{d+1} - \tilde{u})] \\
&\qquad\qquad = -\,\mathrm{sign}[\det(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \tilde{u}, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})]\,.
\end{aligned}$$

By the same calculation we also see that

$$\begin{aligned}
&\mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\beta_i v_i}(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \tilde{u}, \beta_{j+1} v_{j+1}, \ldots, \beta_{i-1} v_{i-1}, 0, \beta_{i+1} v_{i+1}, \ldots, \beta_{d+1} v_{d+1})] \\
&\qquad\qquad = -\,\mathrm{sign}[\det(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \tilde{u}, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})]\,.
\end{aligned}$$

Hence,

$$\begin{aligned}&\mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\tilde{u}}(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, 0, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})]\\ &\qquad = \mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\beta_i v_i}(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \tilde{u}, \beta_{j+1}v_{j+1}, \ldots, \beta_{i-1}v_{i-1}, 0, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})]\,,\end{aligned}$$

and the claim is concluded.

A.3. **Proofs of Proposition 5.1 and Corollary 5.1.** We verify here Proposition 5.1 and Corollary 5.1. We first notice that Corollary 5.1 is an immediate consequence of Proposition 5.1 since whenever $x \in L$ we have that

$$\mathrm{C}_{\mathrm{one}}(\theta, L, x) \subseteq \mathrm{T}_{\mathrm{ube}}(L, \sin(\theta) \cdot r)\,.$$

Proposition 5.1 can be concluded from the following lemma:

**Lemma A.1.** *The set* $\mathrm{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r)$ *can be covered by* $N$ *balls of radius* $2 \cdot \sqrt{2} \cdot \epsilon \cdot r$*, such that*

$$N \leq \frac{(1+\epsilon)^m}{\epsilon^m} \leq \frac{2^m}{\epsilon^m}\,. \tag{81}$$

*Proof.* We choose a set $\{y_i\}_{i=1}^N$ in $\mathrm{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r)$, which is maximally separated by distances $2 \cdot \sqrt{2} \cdot \epsilon \cdot r$. That is,

$$\{y_i\}_{i=1}^N \subseteq \mathrm{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r)\,, \tag{82}$$

$$\|y_i - y_j\| > 2 \cdot \sqrt{2} \cdot \epsilon \cdot r\,, \quad \text{for } 1 \leq i < j \leq N\,, \tag{83}$$

and

$$\mathrm{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r) \subseteq \bigcup_{i=1}^N B(y_i, 2 \cdot \sqrt{2} \cdot \epsilon \cdot r)\,. \tag{84}$$

We denote $z_i := P_L(y_i)$, $i = 1, \ldots, N$, that is, $z_i$ is the projection of the point $y_i$ onto the $m$-dimensional affine plane $L$. Equations (82) and (83) imply that $\{z_i\}_{i=1}^N$ are separated by distances $2 \cdot \epsilon \cdot r$. Consequently, the balls $\{B(z_i, \epsilon \cdot r)\}_{i=1}^N$ are disjoint and $\{z_i\}_{i=1}^N \subseteq L \cap B(x, r)$.

We denote by $\mathcal{H}_m$ the $m$-dimensional Hausdorff measure restricted to $L$, and recall that in our case $\mathcal{H}_m$ is a scaled Lebesgue measure on $L$, such that for any ball $B \subseteq L$, $\mathcal{H}_m(B) = (\mathrm{diam}(B))^m$. We thus obtain that

$$\begin{aligned}N \cdot (2 \cdot r \cdot \epsilon)^m = \sum_{i=1}^N \mathcal{H}_m\left(B\left(z_i, \epsilon \cdot r\right)\right) = \mathcal{H}_m\left(\bigcup_{i=1}^N B\left(z_i, \epsilon \cdot r\right)\right)&\\ \leq \mathcal{H}_m\left(B\left(x, (1+\epsilon) \cdot r\right)\right) = 2^m \cdot (1+\epsilon)^m \cdot r^m\,.&\end{aligned} \tag{85}$$

Equation (81) follows directly from equation (85) and thus the lemma is concluded. □

In order to conclude Proposition 5.1 we note that equation (84) and the definition of an Ahlfors regular measure imply that

$$\mu\left(\mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r)\right) \;\leq\; \sum_{i=1}^N \mu\left(B(y_i, 2 \cdot \sqrt{2} \cdot \epsilon \cdot r)\right) \;\leq\; C_\mu \cdot N \cdot 2^{\frac{3\gamma}{2}} \cdot \epsilon^\gamma \cdot r^\gamma. \tag{86}$$

Then, combining equations (81) and (86), we conclude equation (59).

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